The basic definitions of fuzzy independent set, fuzzy dominating set and fuzzy independent dominating sets are discussed. The aim of this paper is to find on what relations the fuzzy graph has perfect domination number and independent domination number. Finally, the independent domination number for a connected fuzzy graph is obtained.

**Keywords:** fuzzy independent set, fuzzy dominating set, fuzzy independent dominating set, perfect domination number and independent domination number
1. **Fuzzy Independent Set:**

**Definition:**

Let \( G = (\sigma, \mu) \) be a fuzzy graph. Two nodes in a fuzzy graph \( G \) are said to be fuzzy independent if there is no strong arc between them. A subset \( S \) of \( V \) is said to be fuzzy independent set for \( G \) if every two nodes of \( S \) are fuzzy independent.

**Definition:**

Let \( G = (\sigma, \mu) \) be a fuzzy graph. A fuzzy independent set \( S \) of \( G \) is said to be maximal fuzzy independent set if there is no fuzzy independent set whose cardinality is greater than the cardinality of \( S \). The maximum cardinality among all maximal fuzzy independent set is called fuzzy independence number of \( G \) and it is denoted by \( \beta(G) \).

2. **Fuzzy Dominating Set:**

**Definition:**

Let \( G = (\sigma, \mu) \) be a fuzzy graph. A subset \( D \) of \( V \) is said to be a dominating set of \( G \) if for every \( v \in V - D \), there exists a \( u \in D \) such that \( u \) dominates \( v \).

**Definition:**

A dominating set \( D \) of a fuzzy graph \( G \) is called minimal dominating set of \( G \) if there does not exist any dominating set of \( G \), whose cardinality is less than the cardinality of \( D \). Minimum cardinality among all minimum dominating set in \( G \) is called domination number of \( G \) is denoted by \( \gamma(G) \). The smallest cardinality of all independent fuzzy dominating set of \( G \) is called independent fuzzy domination number of \( G \) and is denoted by \( i(G) \).

3. **Perfect Dominating Set:**

**Definition:**

A dominating set \( D \) of a fuzzy graph \( G \) is said to be a perfect dominating set if for each vertex \( v \) not in \( D \), \( v \) is adjacent to exactly one vertex of \( D \).
A perfect dominating set $S$ of a fuzzy graph of $G$ is said to be a minimal perfect dominating set if for each vertex $v$ in $D$, $D - \{v\}$ is not a dominating set.

A perfect dominating set with smallest cardinality is called minimum perfect dominating set. It is called $\gamma_{pf}$ set of $G$.

The cardinality of a minimum perfect dominating set is called the perfect domination number of the fuzzy graph $G$. It is denoted by $\gamma_{pf}(G)$.

### 4. Perfect Domination Number And Independent Domination Number:

**Proposition:**

Let $G = (\sigma, \mu)$ be a fuzzy graph. Let $D$ be a perfect dominating set with the perfect domination number $\gamma_{pf}(G)$ and $i(G)$ denotes the independent domination number. Then clearly $\gamma_{pf}(G) \leq i(G)$.

**Example:**

\[
\begin{align*}
D &= \{b, c\} \quad S = \{a, c\} \\
\gamma_{pf}(G) &= 0.3 + 0.2 = 0.5 \quad i(G) = 0.3 + 0.5 = 0.8 \quad \text{and} \quad \gamma_{pf}(G) \leq i(G).
\end{align*}
\]
Figure 2

D = \{b, c\} S = \{b, f, h, d\}

Here $\gamma_{pf}(G)=0.4+0.5 = 0.9$, $i(G) = 0.4+0.2+0.3+0.3 = 1.2$ and $\gamma_{pf}(G) \leq i(G)$.

**Theorem:**

Let $G = (\sigma, \mu)$ be a fuzzy graph. Let $D$ be a minimum perfect dominating set with the perfect domination number $\gamma_{pf}(G)$. The subgraph $\langle D \rangle$ induced by $D$ has isolated nodes (i.e) $\mu(u, v) = 0$ for all $u, v \in D$ then $\gamma_{pf}(G) = i(G)$ where $i(G)$ denotes the independent domination number.

**Proof:**

It is clear from the definition that the minimum perfect dominating set $D$ is the smallest perfect dominating set among all minimal perfect dominating sets. Since the subgraph induced with the nodes of $D$ are isolated implies that they are independent. Hence $\gamma_{pf}(G) = i(G)$.

In comparing to the crisp case, $\gamma_{pf}(G) = i(G)$ if the graph $G$ is claw free but that is not required for fuzzy graph. Explain this concept in the example given below.

**Example:**
Figure 3

D = \{b, d\}  \ S = \{b, e, f\}

Here \(\gamma_{pf}(G) = 0.3 + 0.2 = 0.5\), \(i(G) = 0.3 + 0.1 + 0.1 = 0.5\) But in crisp case \(\gamma(G) = 2\) and \(i(G) = 3\)

**Corollary:**

Let \(G = (\sigma, \mu)\) be a fuzzy line graph. If the subgraph induced by \(D\) has isolated nodes then \(\gamma_{pf}(L(G)) = i(L(G))\).

**Example:**

\[\begin{array}{c}
D = \{b, c\}  \ S = \{b, d\}
\end{array}\]

Here \(\gamma_{pf}(L(G)) = 0.2 + 0.1 = 0.3\); \(i(L(G)) = 0.2 + 0.3 = 0.5\). Hence \(\gamma_{pf}(L(G)) \neq i(L(G))\).

**Corollary:**

If \(G = (\sigma, \mu)\) is a complete fuzzy graph then \(i(G) < \gamma_{pf}(G)\).

**Proof:**

Since \(G\) is a complete fuzzy graph every arc in \(G\) is a strong arc. Hence \(i(G) = 0\) and \(\gamma(G) = \Lambda\{\sigma(v) \mid \text{for all } v \in V\}\) and \(i(G) = 0\). It is clear that \(i(G) < \gamma_{pf}(G)\).
Example:

\[ D = \{ d \} \quad S = \{ \} \]

Here \( \gamma_{pf}(G) = 0.3 \), \( i(G) = 0 \) and \( i(G) < \gamma_{pf}(G) \).

Example:

\[ D = \{ d \} \quad S = \{ \} \]

Here \( \gamma_{pf}(G) = 0.3 \), \( i(G) = 0 \) and \( i(G) < \gamma_{pf}(G) \).
Definition:

A fuzzy graph $G = (\sigma, \mu)$ is said to be connected if there exists a strongest path between any two nodes of $G$.

Definition:

Let $u$ be a node in fuzzy graph $G$ then $N(u) = \{v : (u, v) \text{ is a strong arc}\}$ is called Neighbourhood of $u$ and $N[u] = N(u) \cup \{u\}$ is called closed neighborhood of $u$. Neighborhood degree of the node is defined by the sum of the weights of the strong neighbor node of $u$ and is denoted by $d_N(u) = \sum_{v \in N(u)} \sigma(v)$

Theorem:

Let $G = (\sigma, \mu)$ be connected fuzzy graph which does not have an induced fuzzy cycle subgraph. Let $D$ be a minimum perfect dominating set with perfect domination number $\gamma_{pf}(G)$. Then $i(G) = \beta(Y) + \sum_{u \in D-Y} d(N(u))$ where $Y$ is maximal independent set of $D$.

Proof:

The nodes in the perfect dominating set $D$ is also connected since the fuzzy graph $G$ is connected. It shows that $\gamma_{pf}(G) \neq i(G)$. Let $Y$ denote the maximal independent set in $D$ and its cardinality is $\beta(Y)$.

The nodes in $V-D$ are independent if not they would have an induced fuzzy cycle which contradicts our assumption. Since the nodes in $D-Y$ are not independent its corresponding neighbors are independent. Hence $i(G) = \beta(Y) + \sum_{u \in D-Y} d(N(u))$

Example:
Figure 7

\[ D = \{a, d, f\} \quad S = \{a, e, f\} \quad Y = \{a, f\} \]

Here \( \gamma_{pf}(G) = 0.6 \quad i(G) = 0.7 \quad \beta(Y) = 0.3 \] and \( i(G) = \beta(Y) + \sum_{u \in D-Y} d(N(u)) \)
References: