Abstract: The purpose of this paper is to prove a common fixed point theorem in a fuzzy metric space using property JCLR concept of weakly compatible mappings. Our main results extends the result of Suneel Kumar, Sunny Chauhan and Chauhan Sunny, Wutiphol Sintunavarat, Poom Kumam[12] and Srinivas V., B.V. B. Reddy, R. Umamaheshwar Rao[15].

Keywords: Fixed Point, self maps Fuzzy metric space, Implicit relation, Weakly compatible maps, Property (CLRg), Property (JCLR) and fixed point.
1. Introduction:

The concept of the fuzzy metric spaces has been introduced by Zadeh L.A.[6], in the recent past many authors developed the theory of fuzzy sets and applications. Implicit relations and CLRg property are used as a tool for finding common fixed point of contraction maps. Recently, Pant V. [14] proved a common fixed point theorem without completeness of space and continuity of involved mappings in FM-space, which generalizes the results of Singh B. and Jain [2]. In this paper, we prove a common fixed point theorem for six-maps in FM-space satisfying contractive type implicit relations.


2. Preliminaries

Definition 2.1. A binary operation */ : [0,1] x [0,1] ⟷ [0,1] is called a continuous t-norm if

([0,1],*) satisfies the following conditions:

(i) * is commutative and associative;
(ii) * is continuous;
(iii) a*1 = a; ∀ a∈[0,1]
(iv) a*b ≤ c*d whenever a ≤ c and b ≤ d . ∀ a, b, c, d ∈ [0,1].

Definition 2.2. The triplet (X, M,*) is a FM-space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in X x [0,∞) satisfying the following conditions for all x,y,z ∈ X and t, s >0,

(FM-1) M(x,y,0) = 0;
(FM-2) M(x,y,t) = 1 for all t >0 if and only if x= y;
(FM-3) M(x,y,t) = M(y,x,t);
(FM-4) M(x,y,t)*M(y,z,s) ≤ M(x,z,t+s);
(FM-5) M(x,y,·) : [0,∞) ⟷ [0,1] is left continuous.

The function M(x,y,t) denote the degree of nearness between x and y with respect to ‘t’.

We observe M(x,y,t) = 1 when x = y for all t>0 and M(x,y,t) = 0 with t = ∞. Some of the properties of fuzzy metric space and examples are given in paper George A. and Veeramani [1].
Example 2.1. Let \((X, d)\) be a metric space. Define \(a*b = \min\{a,b\}\) for all \(x,y \in X\) and \(t > 0\), 
\[
M(x,y,t) = \frac{t}{t + d(x,y)}.
\]
Then \((X, M,*)\) is a FM space and the fuzzy metric \(M\) induced by the metric \(d\) is often referred to as the standard fuzzy metric. From the above example every metric induces a fuzzy metric but there exist no metric on \(X\) satisfying.

Definition 2.3. ([7]) Let \((X,M,*)\) be a FM-space. Then

[1] A sequence \(\{x_n\}\) in \(X\) is said to be convergent to a point \(x \in X\) (denoted by \(\lim_{n \to \infty} x_n = x\)) if 
\[
\lim_{n \to \infty} M(x_n, x, t) = 1. \forall t > 0.
\]

[2] A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if \(\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1. \forall t > 0\) and \(p > 0\).

[3] A FM-space in which every Cauchy sequence is convergent is called complete.

Lemma 2.1. [13] Let \((X,M,*)\) be a FM-space for all \(x,y \in X\), \(t > 0\) and if for a number \(k \in (0,1)\), 
\[
M(x,y,kt) \geq M(x,y,t) \forall x, y \in X \text{ and } t > 0.
\]

Proposition 2.1. In the fuzzy metric space \((X,M,*)\) if \(a*b \geq a\) for all \(a \in [0,1]\) then \(a*b = \min\{a,b\}\).

Definition 2.4. ([11]) Let \(A\) and \(S\) maps from a FM-space \((X,M,*)\) into itself. The maps \(A\) and \(S\) are said to be compatible (or asymptotically commuting), if for all \(t\), 
\[
\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1.
\]
Whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Ax_n = z = \lim_{n \to \infty} Sx_n\) for some \(z \in X\).

Definition 2.5. ([2]) Let \(A\) and \(S\) be maps from a FM-space \((X,M,*)\) into itself.
The maps are said to be weakly compatible if they commute at their coincidence points, i.e. 
\(Az = Sz\) implies that \(ASz = SAz\).
It is clear that every compatible pair is weakly compatible but its converse need not be true.
To know the relation among commutatively, compatibility and weakly compatibility refer some of the research papers like Jungek [3], Pant[14], Srinivas and others[15].
Definition 2.6. ([8]) A pair \((f,g)\) of self mappings of a fuzzy metric space \((X,M,\ast)\) is said to satisfy the “common limit in the range of \(g\)” property (CLR\(g\)) if there exists a sequence \(\{x_n\}\) in \(X\) such that
\[
\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = g u, \text{ for some } u \in X.
\]

Example 2.2. Let \((X,M,\ast)\) be a fuzzy metric space with \(X = [0,\infty)\) and
\[
M(x,y,t) = \begin{cases} 
\frac{t}{t + |x-y|}, & \text{if } t > 0; \\
0, & \text{if } t > 0.
\end{cases} \quad \forall \ x,y \in X.
\]
Define self mappings \(f\) and \(g\) on \(X\) by \(f(x) = x+2\) and \(g(x) = 3x\), \(\forall \ x \in X\). Let a sequence \(\{x_n\} = \{\frac{1}{n}\}n \in \mathbb{N}\) in \(X\) we have,
\[
\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = 3 = g(1) \in X, \text{ which shows that } f \text{ and } g \text{ satisfy the (CLR}\(g\)\) property.
\]

Definition 2.7. ([12]) Let \((X,M,\ast)\) be a fuzzy metric space and \(A,S,B,T : X \to X\). The pair \((A,S)\) and \((B,T)\) are said to satisfy the “Joint common limit in the range” property (JCLR property) if there exists a sequence \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[
Ax_n = Sx_n = By_n = Ty_n = Su = Tu, \text{ for some } u \in X. \quad \ldots(1)
\]

Remark 2.1. If \(B = A\), \(T = S\) and \(\{x_n\} = \{y_n\}\) in \((1)\), then we get the definition of (CLR\(g\)).

3. Main Results:

Theorem 3.1 Let \(A,B,S\) and \(T\) be self maps of a FM-space \((X,M,\ast)\) satisfying
\[
(3.1.1) \ (A,S) \text{ or } (B,T) \text{ satisfies the property (JCLR)};
\]
\[
(3.1.2) \ [M(Ax , By, kt)]^2 + M(Ax , By, kt) M(Ty, Sx, kt) \geq \{K_1[M(By, Sx, kt) + M(Ax , Ty, kt)] + K_2[M(Ax , Sx, kt) + M(By , Ty, kt)]\}M(Ty , Sx, kt).
\]
Where for all \(x,y \in X\) and \(k_1, k_2 \geq 0\), \(k_1 + k_2 \geq 1\).
\[
(3.1.3) \ A(X) \subseteq T(X), \ B(X) \subseteq S(X);
\]
\[
(3.1.4) \text{One of } A(X), B(X), S(X) \text{ and } T(X) \text{ is a complete subspace of } X.
\]
Then the pairs \((A,S)\) and \((B,T)\) have a point of coincidence each. Moreover, \(A,\ B,\ S,\ \text{and } T\) have a unique common fixed point provided the pairs \((A,S)\) and \((B,T)\) commute pairwise.
(i.e. \(A S = SA\) and \(B T = TB\)).

Proof: Since the pair \((A,S)\) and \((B,T)\) satisfies the property (JCLR), then there exists a sequence \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that,
\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = Su = Tu, \text{ for some } u \in X. \]

Now we show that \( Tu = Bu \). Using (3.1.2), with putting \( x = x_n \) and \( y = u \).

\[
[M(Ax_n, Bu, kt)]^2 * M(Ax_n, Bu, kt)M(Tu, Sx_n, ku) \geq \{k_1[M(Bu, Sx_n, ku) * M(Ax_n, Tu, ku)] \\
+ k_2[M(Ax_n, Sx_n, ku) * M(Bu, Tu, ku)])M(Tu, Sx_n, ku); \\
[M(Tu, Bu, kt)]^2 * M(Tu, Bu, kt)M(Tu, Tu, ku) \geq \{k_1[M(Bu, Tu, ku) * M(Tu, Tu, ku)] \\
+ k_2[M(Tu, Tu, ku) * M(Bu, Tu, ku)])M(Tu, Tu, ku); \\
[M(Tu, Bu, kt)]^2 * M(Tu, Bu, ku) \geq \{k_1M(Bu, Tu, ku) \\
+ k_2M(Bu, Tu, ku)); \\
[M(Bu, Tu, ku)] \geq \{k_1+k_2); \\
[M(Bu, Tu, ku)] \geq 1; \]
\]

Therefore
\[ Bu = Tu. \]

Since \( k_1 + k_2 \geq 1 \), where \( k_1, k_2 \geq 0 \). Thus \( Tu = Bu \).

Next we show that \( Au = Tu \). Using (3.1.2), with putting \( x = u \) and \( y = y_n \).

\[
[M(Au, By_n, ku)]^2 * M(Au, By_n, ku)M(Tu, Su, ku) \geq \{k_1[M(By_n, Su, ku) * M(Au, Ty_n, ku)] \\
+ k_2[M(Au, Su, ku) * M(By_n, Ty_n, ku)])M(Tu, Su, ku); \\
[M(Au, Tu, ku)]^2 * M(Au, Tu, ku)M(Tu, Su, ku) \geq \{k_1M(Tu, Su, ku) * M(Au, Tu, ku)] \\
+ k_2[M(Au, Tu, ku) * M(Tu, Tu, ku)])M(Tu, Su, ku); \\
[M(Au, Tu, ku)]^2 * M(Au, Tu, ku)M(Su, Su, ku) \geq \{k_1[M(Su, Su, ku) * M(Au, Tu, ku)] \\
+ k_2[M(Au, Tu, ku) * M(Tu, Tu, ku)])M(Su, Su, ku); \\
[M(Au, Tu, ku)]^2 * M(Au, Tu, ku) 1 \geq \{k_1[1 * M(Au, Tu, ku)] \\
+ k_2[M(Au, Tu, ku) * 1]; \\
[M(Au, Tu, ku)] \geq \{k_1[M(Au, Tu, ku) + 1]M(Au, Tu, ku)] \\
M(Au, Tu, ku) \geq \{k_1+k_2); \\
M(Au, Tu, ku) \geq 1; \]
\]

Therefore
\[ Au = Tu. \]

Now assume that \( z = Au = Bu = Tu = Su \). Since the pair \( (A, S) \) is commute pairwise

\[ ASu = SAu \Rightarrow Az = Sz. \]

Similarly the pair \( (B, T) \) is commute pairwise also then,

\[ BTu = TBu \Rightarrow Bz = Tz. \]

Now we have to show that \( Az = z \). Using (3.1.2), with putting \( x = z \) and \( y = u \).

\[
[M(Az , Bu, kt)]^2 * M(Az , Bu, kt) M(Tu, Sz, ku) \geq \{K_1[M(Bu, Sz, ku) * M(Az , Tu, ku)] \\
+ K_2[M(Az , Sx_n, ku) * M(Bu, Tu, ku)])M(Tu, Sz, ku); \\
[M(Tu, Bu, kt)]^2 * M(Tu, Bu, ku)M(Tu, Tu, ku) \geq \{K_1[M(Bu, Tu, ku) * M(Tu, Tu, ku)] \\
+ K_2[M(Tu, Tu, ku) * M(Bu, Tu, ku)])M(Tu, Tu, ku); \\
[M(Tu, Bu, ku)]^2 * M(Tu, Bu, ku) \geq \{K_1M(Bu, Tu, ku) \\
+ K_2M(Bu, Tu, ku)); \\
[M(Bu, Tu, ku)] \geq \{K_1+K_2); \\
[M(Bu, Tu, ku)] \geq 1; \]
\]

Therefore
\[ Au = Tu. \]
\[ +K_2[M(Az, Sz, kt) \ast M(Bu, Tu, kt)]M(Tu, Sz, kt); \]
\[ [M(Az, z, kt)]^2 \ast M(Az, z, kt) M(z, Az, kt) \geq [K_1[M(z, Az, kt) \ast M(Az, z, kt)] \]
\[ +K_2[M(Az, Az, kt) \ast M(z, z, kt)]M(z, Az, kt); \]
\[ [M(Az, z, kt)]^2 \ast M(Az, z, kt) M(z, Az, kt) \geq [K_1[M(z, Az, kt) \ast M(Az, z, kt)] \]
\[ +K_2[1 \ast 1]M(z, Az, kt); \]
\[ [M(Az, z, kt)]^2 \ast M(Az, z, kt) M(z, Az, kt) \geq [K_1[M(z, Az, kt) \ast M(Az, z, kt)] \]
\[ +K_2[1]M(z, Az, kt); \]
\[ [M(Az, z, kt)]^2 \ast M(z, Az, kt) \geq [K_1[M(z, Az, kt) + K_2]M(z, Az, kt); \]
\[ M(Az, z, kt) \geq [K_1[M(z, Az, kt)] + K_2]; \]
\[ M(z, Az, kt)(1 - k_1) \geq K_2; \]
\[ [M(Az, z, kt)] \geq \frac{k_2}{1 - k_1}; \]

Since \( k_1 + k_2 \geq 1 \)
\[ [M(Az, z, kt)] \geq 1; \]

Therefore \( Az = z. \)

Now we have to show that \( Bz = z. \) Using (3.1.2), with Put \( x = u \) and \( y = z. \)
\[ [M(Au, Bz, kt)]^2 \ast M(Au, Bz, kt) M(Tz, Su, kt) \geq [K_1[M(Bz, Su, kt) \ast M(Au, Tz, kt)] \]
\[ +K_2[M(Au, Su, kt) \ast M(Bz, Tz, kt)]M(Tz, Su, kt); \]
\[ [M(z, Bz, kt)]^2 \ast M(z, Bz, kt) M(z, Bz, kt) \geq [K_1[M(z, Bz, kt) \ast M(z, Bz, kt)] \]
\[ +K_2[M(z, z, kt) \ast M(z, Bz, kt)]M(Bz, z, kt); \]
\[ [M(z, Bz, kt)]^2 \ast [M(z, Bz, kt)]^2 \geq [K_1[M(z, Bz, z, kt)] \]
\[ +K_2[1 \ast 1]M(Bz, z, kt); \]
\[ [M(z, Bz, kt)]^2 \geq [K_1[M(z, Bz, z, kt)] \]
\[ +K_2]M(Bz, z, kt); \]
\[ [M(z, Bz, kt)] \geq K_1[M(z, Bz, z, kt)] \geq k_2; \]
\[ M(z, Bz, kt)(1 - k_1) \geq K_2; \]
\[ [M(z, Bz, kt)] \geq \frac{k_2}{1 - k_1}; \]

Since \( k_1 + k_2 \geq 1 \)
\[ [M(z, Bz, kt)] \geq 1; \]

Therefore \( Bz = z. \)

Since \( Az = Bz = Sz = Tz = z, \) we get \( z \) is a common fixed point of \( A, B, S \) and \( T. \) The uniqueness of the fixed point can be easily proved.

**Example:** Let \( X = [0,1] \) \( M(x,y,t) = \frac{t}{t + d(x,y)} \), where \( d(x,y) = |x - y| \)

\[ A x = B x = \begin{cases} \frac{1}{16} & \text{if} \ 0 \leq x \leq \frac{1}{16} \\ \frac{1}{12} & \text{if} \ \frac{1}{16} < x \leq 1 \end{cases} \]
\[ S x = T x = \begin{cases} \frac{1}{8} - x & \text{if} \ 0 \leq x \leq \frac{1}{16} \\ \frac{1}{8} & \text{if} \ \frac{1}{16} < x \leq 1 \end{cases} \]
Then $A(X) = B(X) = \left\{ \frac{1}{16}, \frac{1}{12} \right\}$ while $S(X) = T(X) = \left\{ \frac{1}{8} \right\} \cup \left\{ \frac{1}{16} \right\}$ so that the conditions $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ are satisfied. From the example given above, clearly the pairs $(A,S)$ and $(B,T)$ are weakly compatible as they commute at coincident point $\frac{1}{16}$. But the pairs $(A,S)$ and $(B,T)$ are not compatible. For this, take a sequence $x_n = \left( \frac{1}{16} \right)$ for $n \geq 1$,

Then $\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \frac{1}{16}$ and $\lim_{n \to \infty} A S x_n = \frac{1}{12}$ also $\lim_{n \to \infty} S A x_n = \frac{1}{16}$.

So that $\lim_{n \to \infty} M(ASx_n, SAx_n, t) \neq 1$.

Also note that none of the mappings are continuous and the rational inequality holds for the values of $0 \leq k_1 + k_2 \leq 1$, where $k_1, k_2 \geq 0$. Clearly $\frac{1}{16}$ is the unique common fixed point of $A, B, S$ and $T$.

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