A Note On The Splitcommon Fixed-Point Problem For Strongly Quasi-Nonexpansive Operator In Hilbert Space

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Abstract: Based on the recent work by Censor and Segal (2009 J. Convex Anal.16), and inspired by Moudafi (2010 Inverse Problem 26), in this paper, we study the modified algorithm of Yu and Sheng for the strongly quasi-nonexpansive operators to solve the split common fixed-point problem (SCFP) in a Hilbert space. Our results extend and improved/developed some recent result announced.

Keywords: Convex feasibility, Split feasibility, Split common fixed point, strongly quasi-nonexpansive operator, Iterative algorithm.
1. Introduction:

Let $H_1$ and $H_2$ be a Hilbert spaces, $A: H_1 \to H_2$ be a bounded linear operator and $A^*$ be an adjoint of $A$. Given integer’s $p$, $r \geq 1$ and also given sequence of nonempty, closed, convex subsets $\{C_i\}_i^p$ and $\{Q_j\}_j^r$ of $H_1$ and $H_2$ respectively, the convex feasibility problem (CFP) is formulated as finding a point $x^* \in H_1$ satisfying the property:

$$x^* \in \bigcap_i^p C_i. \quad (1.1)$$

Note that, CFP (1.1) has received a lot of attention due to its extensive applications in many applied disciplines, diverse as approximation theorem, image recovery, signal processing, control theory, biomedical engineering, communication and geophysics.

The multiple set split feasibility problem (MSSFP) was recently introduced and studied by Censor, Elfving, Kopf, and Bortfeld, and is formulated as finding a point $x^* \in H_1$ with the property:

$$x^* \in \bigcap_i^p C_i \text{ and } Ax^* \in \bigcap_j^r Q_j. \quad (1.2)$$

If in a MSSFP (1.2) $p = r = 1$, we get what is called the split feasibility problem (SFP), which is formulated as finding a point $x^* \in H_1$ with the property:

$$x^* \in C \text{ and } Ax^* \in Q. \quad (1.3)$$

Where $C$ and $Q$ are nonempty, closed and convex subsets of $H_1$ and $H_2$ respectively.

Note that, SFP (1.3) and MSSFP (1.2) model image retrieval and intensity modulated radiation therapy and have recently been studied by many Researchers and references therein.

The MSSFP (1.2) can be viewed as a special case of the CFP (1.1) since (1.2) can be rewriting as

$$x^* \in \bigcap_i^{p+r} C_i, \quad C_{p+j} = \{x^* \in H: x^* \in A^{-1}(Q_j), \ 1 \leq j \leq r\}. \quad (1.4)$$

However, the methodologies for studying the MSSFP (1.2) are actually different from those for the CFP (1.1) in order to avoid usage of the inverse of $A$. In other word, the method for
solving CFP (1.1) may not apply to solve MSSFP (1.2) straight forwardly without involving the inverse of A. The CQ algorithm of Byne is such an example where only the operator of A is used without involving the inverse.

Since every closed convex subset of Hilbert space is the fixed point set of its associating projection, the CFP (1.1) becomes a special case of the common fixed-point problem (CFPP) of finding a point \( x^* \in H_1 \) with property:

\[
x^* \in \bigcap_{i=1}^{p} \text{Fix}(T_i).
\]  

(1.4)

Where each, \( T_i: H_1 \to H_2 \) are some (nonlinear) mapping. Similarly the MSSFP (1.2) becomes a special case of the split common fixed point problem (SCFPP) of finding a point \( x^* \in H_1 \) with the property:

\[
x^* \in \bigcap_{i=1}^{p} \text{Fix}(U_i) \text{ and } Ax^* \in \bigcap_{j=1}^{r} \text{Fix}(T_j)
\]  

(1.5)

Where each, \( U_i: H_1 \to H_1 \) (i=1, 2, 3... p) and \( T_j: H_2 \to H_2 \) (j=1, 2, 3, ..., r) are some nonlinear operators.

If \( p = r = 1 \), problem (1.5) is reduces to find a point \( x^* \in H_1 \) with property:

\[
x^* \in \text{Fix}(U) \text{ and } Ax^* \in \text{Fix}(T)
\]  

(1.6)

This is usually called the two-set SCFPP.

The concept of SCFPP in finite dimensional Hilbert space was first introduce by Censor and Segal who invented an algorithm of the two-set SCFPP which generate a sequence \( \{x_n\} \) according to the following iterative procedure:

\[
x_{n+1} = U(x_n + \gamma A^*(T - I)Ax_n), \quad n \geq 0.
\]  

(1.7)

Where the initial guess \( x_0 \in H \) is choosing arbitrarily and \( 0 < \gamma \leq \frac{1}{\|A\|^2} \).

Inspired by the work of Censor and Segal, Moudafi introduced the following algorithm for \( \mu \)-demicontactive operator in Hilbert space:

\[
\begin{align*}
\{ u_n &= x_n + \gamma A^*(T - I)Ax_n \\
x_{n+1} &= (1 - t_n)u_n + t_nU(u_n), \quad n \geq 0
\end{align*}
\]

Where \( \gamma \in \left(0, \frac{(1-\mu)}{\lambda}\right) \) with \( \lambda \) the being spectral radius of the operator \( A^*A \) and
Using fejer-monotone and the demi closed properties of \((I - U)\) and \((I - T)\) at origin, in 2010 Moudafi proved convergence theorem based on the work of Censor and Segal. And also in 2011 Moudafi, Sheng and Chen gave their result of pseudo-demi contractive operators for the split common fixed-point problems. In 2012, Yu and Sheng that modified the algorithm proposed by Moudafi and they extend the operator to the class of firmly pseudo-demi contractive operator. In this paper, we study the modified algorithm of Yu and Sheng and we used the strongly quasi nonexpansive operator to obtain the weak convergence result of SCFPP (1.5).

2. Preliminaries:

i. Throughout this paper, we adopt the notation:

ii. \(I:\) the identity operator on Hilbert space \(H\).

iii. \(\text{Fix}(T):\) the set of fixed point of an operator \(T:H \rightarrow H\)

iv. \(\Omega:\) The solution set of SCFPP (1.5).

v. \(\omega_{\omega}(x_{n})\): The set of the cluster point of \(x_{n}\) in the weak topology i.e.

\[\{\exists x_{n_{j}} \text{ of } x_{n} \text{ such that } x_{n_{j}} \rightharpoonup x\}\]

vi. \(x_{n} \rightharpoonup x; \{x_{n}\}\) Converge in norm to \(x\)

vii. \(x_{n} \rightharpoonup x; \{x_{n}\}\) Converge weakly to \(x\)

**Definition 2.1:** Assume that \(C\) is a closed convex nonempty subset of a real Hilbert space \(H\). A sequence \(\{x_{n}\}\) in \(H\) is said to be Fejer monotone with respect to \(C\) if and only if

\[\|x_{n+1} - z\| \leq \|x_{n} - z\|, \quad \text{for all } n \geq 1 \text{ and } z \in C\]

**Definition 2.2:** Let \(T:H \rightarrow H\) be an operator. We say that \((I - T)\) is demi closed at zero, if for any sequence \(x_{n}\) in \(H\), there holds the following implication: \(x_{n} \rightharpoonup x\) and \((I - T)x_{n} \rightarrow 0\) as \(n \rightarrow \infty\), then \((I - T)x = 0\).

**Definition 2.3:** A Banach space \(E\) has Kadec-Klee property, if for every sequence \(x_{n} \in E\) such that \(x_{n} \rightharpoonup x\) and

\[\|x_{n}\| \rightarrow \|x\|,\text{ then } x_{n} \rightharpoonup xn \rightarrow \infty\]
**Definition 2.4**: An operator $T : H \to H$ is said to be

i. **nonexpansive** if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in H$

ii. **quasi-nonexpansive** if $\text{Fix}(T) \neq \emptyset$ and $\|Tx - z\| \leq \|x - z\|$, for all $x \in H$ and $z \in \text{Fix}(T)$

iii. **strictly quasi-nonexpansive** if $\text{Fix}(T) \neq \emptyset$ and $\|Tx - z\| < \|x - z\|$, for all $x \in H/\text{Fix}(T)$ and $z \in \text{Fix}(T)$

iv. **$\alpha$-strongly quasi-nonexpansive** if there exist $\alpha > 0$ with the property: $\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|x - Tx\|^2$, for all $x \in H$ and $z \in \text{Fix}(T)$.

This is an equivalent to

$$\langle x - z, Tx - x \rangle \leq -\frac{1-\alpha}{2}\|x - Tx\|^2$$

for all $x \in H$ and $z \in \text{Fix}(T)$.

**Definition 2.5**: An operator $T : H \to H$ is said to be:

i. **Demi contractive**; if there exist a constant $\beta < 1$ such that $\|Tx - z\|^2 \leq \|x - z\|^2 + \beta\|x - Tx\|^2$, for all $x \in H$ and $z \in \text{Fix}(T)$.

ii. **Pseudo-demi contractive**; if there exist a constant $\alpha > 1$ such that $\|Tx - z\|^2 \leq \alpha\|x - z\|^2 + \|x - Tx\|^2$, for all $x \in H$ and $z \in \text{Fix}(T)$.

iii. **Firmly pseudo-demi contractive**; if there exist constants $\alpha > 1$ and $\beta > 1$ such that $\|Tx - z\|^2 \leq \alpha\|x - z\|^2 + \beta\|x - Tx\|^2$, for all $x \in H$ and $z \in \text{Fix}(T)$.

**Lemma 2.6**: Let $T : H \to H$ be an operator. Then the following statements are equivalent

1. $T$ is class $\tau$ operator;

2. $\|x - Tx\|^2 \leq \langle x - z, x - Tx \rangle$, for all $x \in H$ and $z \in \text{Fix}(T)$.

3. there hold the relation: $\|Tx - z\|^2 \leq \|x - z\|^2 - \|x - Tx\|^2$, for all $x \in H$ and $z \in \text{Fix}(T)$.

Consequently a class $\tau$ operator is 1 - strongly quasi - nonexpansive.

**Lemma 2.7**: If a sequence $\{x_n\}$ is fejer monotone with respect to a closed convex nonempty subset $C$, then the following hold.

i. $x_n \rightharpoonup x$ if and if $\omega_{\omega}(x_n) \subseteq C$;
ii. The sequence \( \{ P_{\Omega}x_n \} \) converges strongly to some point in \( C \);

iii. if \( x_n \to x \in C \), Then \( x = \lim_{n \to \infty} P_{\Omega}x_n \).

**Lemma 2.8:** Let \( H \) be a Hilbert space and let \( \{ x_n \} \) be a sequence in \( H \) such that there exist a nonempty set \( S \subseteq H \) satisfying the following:

i. For every sequence \( x \in H, \lim_{n \to \infty} \| x_n - x \| \) exist;

ii. Any weak-cluster point of the sequence \( \{ x_n \} \) belongs in \( S \). Then there exist \( x \) in \( S \) such that \( \{ x_n \} \) weakly converges to \( x \).

3. **Main Results:**

In what follows, we will focus our attention on the following general two-operator split common fixed-point problem: find

\[ x^* \in C \text{ and } Ax^* \in Q. \]  \( (3.1) \)

Where \( A: H_1 \to H_2 \) is bounded and linear operator, \( U: H_1 \to H_1 \) and \( T: H_2 \to H_2 \) are two strongly quasi-nonexpansive operators with nonempty fixed-point set \( \text{Fix} (U) = C \) and \( \text{Fix} (T) = Q \),

\[ \| Tx - z \|^2 \leq \| x - z \|^2 - \alpha \| x - Tx \|^2, \text{for all} \]
\[ x \in H \text{ and } z \in \text{Fix}(T). \]  \( (3.2) \)

\[ \| Tx - z \|^2 \leq \| x - z \|^2 - \beta \| x - Tx \|^2, \text{for all} \]
\[ x \in H \text{ and } z \in \text{Fix}(T). \]  \( (3.3) \)

And denote the solution set of the two-operator SCFPP by

\[ \Omega = \{ x^* \in C \text{ and } Ax^* \in Q \}. \]  \( (3.4) \)

Based on the algorithm of , we have the following algorithm to solve \( (3.1) \)

\[
\begin{align*}
    u_k &= x_k + \gamma A^* (T - I)Ax_k \\
    x_{k+1} &= (1 - t_k)u_k + t_k U(u_k), \quad k \geq 0
\end{align*}
\]  \( (3.5) \)

Where \( \frac{1 - \beta}{\lambda} < \gamma < 0 \) with \( \lambda \) being the spectral radius of the operator \( A^*A \) and \( t_k > 0 \) and \( x_0 \in H \) is choosing arbitrarily.
Theorem 3.1: Let $H_1 \rightarrow H_2$ be a bounded linear operator, $U: H_1 \rightarrow H_1$ and $T: H_2 \rightarrow H_2$ be two strongly quasi-nonexpansive operators with $\text{Fix}(U) = C$ and $\text{Fix}(T) = Q$. Assume that $(U - I)$ and $(T - I)$ are both demi-closed at zero, if $\Omega$ is nonempty, then any sequence $\{x_k\}$ generated by algorithm (3.5) converges weakly to a split common fixed point $x^* \in \Omega$.

Proof. As we are in Hilbert space. Now, taking $x^* \in \Omega$ that is $x^* \in \text{Fix}(U)$ and $Ax^* \in \text{Fix}(T)$ and by definition (2.4 (d)) we deduce that that

$$
\|x_{k+1} - x^*\|^2 = \|(1 - t_k)u_k + t_k U(u_k) - x^*\|^2
$$

$$
= \|u_k - x^*\|^2 + 2t_k \langle u_k - x^*, Uu_k - u_k \rangle + t_k^2 \|Uu_k - u_k\|^2
$$

$$
\leq \|u_k - x^*\|^2 - t_k (1 + \alpha) \|Uu_k - u_k\|^2 + t_k^2 \|Uu_k - u_k\|^2
$$

$$
\leq \|u_k - x^*\|^2 - t_k (1 + \alpha - t_k) \|Uu_k - u_k\|^2
$$

$$
\Rightarrow \|x_{k+1} - x^*\|^2 \leq \|u_k - x^*\|^2 - t_k (1 + \alpha - t_k) \|Uu_k - u_k\|^2 \quad (3.6).
$$

On the other hand, we have

$$
\|u_k - x^*\|^2 = \|x_k + \gamma A^*(T - I)Ax_k - x^*\|^2
$$

$$
= \|x_k - x^*\|^2 + 2\gamma \langle x_k - x^*, A^*(T - I)Ax_k \rangle + \gamma^2 \|A^*(T - I)Ax_k\|^2
$$

$$
\leq \|x_k - x^*\|^2 - \gamma (1 + \beta - \gamma \lambda) \|(T - I)Ax_k\|^2
$$

$$
\Rightarrow \|u_k - x^*\|^2 \leq \|x_k - x^*\|^2 - \gamma (1 + \beta - \gamma \lambda) \|(T - I)Ax_k\|^2.
$$

(3.7)

Substituting (3.7) into (3.6), we get the following:
\[ \|x_{k+1} - x^*\| \leq \|x_k - x^*\|^2 - \gamma (1 + \beta - \gamma \lambda)\|(T - I)Ax_k\|^2 - t_k(1 + \alpha - t_k)\|Uu_k - u_k\|^2 \quad (3.8) \]

Since \( \gamma < 0; \beta > 0; \alpha > 0; \lambda > 0 \) and \( t_k > 0 \), we obtain that \(-\gamma (1 + \beta - \gamma \lambda) < 0 \) and 
\(-t_k(1 + \alpha - t_k) < 0 \)

then from equation (3.8), we deduce that \( \{x_k\} \) is a Fejér monotone and moreover
\( \{\|x_{k+1} - x^*\|\}_{k \in \mathbb{N}} \) is a monotonically decreasing sequence, hence converges. Therefore we have
\[ \lim_{k \to \infty} \|(T - I)Ax_k\| = 0. \quad (3.9) \]

From the Fejér monotonicity of \( \{x_k\} \), it follows that the sequence is bounded. Denoting
by \( x^* \) a weak cluster point of \( \{x_k\} \), let \( j=0,1,2,... \) be the sequence of indices, such that
\[ w = \lim_{\nu \to \infty} x_{k_{\nu}} = x^*. \quad (3.10) \]

Then, from (3.8) and the demi closeness of \( (T - I) \) at zero, we obtain
\[ T(Ax^*) = Ax^*. \quad (3.11) \]

From which it follows \( Ax^* \in Q \). from (3.3), by considering \( u_k = x_k + \gamma A^*(T - I)Ax_k \), it follows that
\[ w = \lim_{\nu \to \infty} u_{k_{\nu}} = x^*. \quad (3.12) \]

Again from (3.8) and the convergence of the sequence \( \{\|x_{k+1} - x^*\|\}_{k \in \mathbb{N}} \), we also have
\[ \lim_{k \to \infty} \|(U - I)u_k\| = 0. \quad (3.13) \]

Which, combined with the demi closeness of \( (U - I) \) at zero and weak convergence
of \( \{u_{k_{\nu}}\} \) to \( x^* \), yields
\[ Ux^* = x^*. \quad (3.14) \]

Hence \( x^* \in C \), and therefore \( x^* \in \Omega \). Since there is no more than one weak-cluster point,
the weak convergence of the whole sequence \( x_k \) follows by applying Lemma (2.8)
with \( S = \Omega \). i.e.
\[ x_k \rightharpoonup x^* \]
4. Conclusion:

In this paper, the modified algorithm of Yuand Sheng was studied for a class of strongly quasi nonexpansive operators to solve the split common fixed-point (1.5) and some beautiful lemmas was used to prove the weak convergence of the modified algorithm. Our result extends and improved some recent result announced.

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Reference:


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